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On the solutions of a second order difference equation

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Abstract. In this paper, we discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1}}{a x_n + b x_{n-1}}, \quad n \in \mathbb{N}_0,$$

where a, b are real numbers and the initial conditions x_{-1}, x_0 are real numbers.

We determine the forbidden set and give an explicit formula for the solutions. We show the existence of periodic solutions, under certain conditions.

1. Introduction

In this paper, we discuss the global behavior of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1}}{a x_n + b x_{n-1}}, \quad n \in \mathbb{N}, \tag{1}$$

where a, b are real numbers and the initial conditions x_{-1}, x_0 are real numbers. Results concerning rational difference equations having quadratic terms are included in some publications such as [1]-[21] and the references cited therein.

Using the substitution $y_n = \frac{1}{x_n}$, we can obtain the linear second order homogeneous difference equation

$$y_{n+1} = b y_n + a y_{n-1}, \quad n \in \mathbb{N} \tag{2}$$

The characteristic equation of equation (2) is

$$\lambda^2 - b\lambda - a = 0. \tag{3}$$

Equation (3) has two roots $\lambda_- = \frac{b}{2} - \frac{\sqrt{b^2+4a}}{2}$ and $\lambda_+ = \frac{b}{2} + \frac{\sqrt{b^2+4a}}{2}$. The form of the solution should be according to the value of the quantity $b^2 + 4a$.

The following theorem [12] is useful in studying the solutions of the difference equation (2).

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Theorem 1.1. *The following statements holds:*

(1) *All solutions of (2) oscillates (about zero) if and only if the characteristic equation has no positive roots.*
(2) *All solutions of (2) converge to zero if and only if* $max\{\lambda_1, \lambda_2\} < 1$.

In order to study the solutions of the difference equation (1), we consider the two cases:

- Case $ab > 0$;
- Case $ab < 0$.

The case $ab = 0$ reduces equation (1) to the first order homogeneous difference equation $x_{n+1} = \frac{1}{b}x_n$ when $a = 0$, and to the second order homogeneous difference equation $x_{n+1} = \frac{1}{a}x_{n-1}$ when $b = 0$, which are easy to investigate.

2. CASE $ab>0$

In this section we discuss the behavior of the solutions when $ab > 0$.

2.1. **Case** $a > 0$**,** $b > 0$**.** In this case, we have $b^2 + 4a > 0$ and so the roots λ_- and λ_+ are real such that $\lambda_- < 0 < \frac{b}{2} < \lambda_+$. The solution of equation (2) is

$$y_n = c_1\lambda_-^n + c_2\lambda_+^n, \quad n = -1, 0, 1, ...$$

Using the initials y_{-1} and y_0, the values of c_1 and c_2 are

$$c_1 = \frac{a(y_0 - y_{-1}\lambda_+)}{(\lambda_+ - \lambda_-)\lambda_+} \quad \text{and} \quad c_2 = \frac{a(y_{-1}\lambda_- - y_0)}{(\lambda_+ - \lambda_-)\lambda_-}.$$

Then,

$$y_n = \frac{1}{\lambda_+ - \lambda_-}[y_0(\lambda_+^{n+1} - \lambda_-^{n+1}) + ay_{-1}(\lambda_+^n - \lambda_-^n)], \quad n = -1, 0, 1, ...$$

If we put $y_n = 0$, we get

$$y_0 = -a\left(\frac{\lambda_+^n - \lambda_-^n}{\lambda_+^{n+1} - \lambda_-^{n+1}}\right)y_{-1}.$$

Hence, we conclude that, the forbidden set of equation (1) is

$$F = \bigcup_{n=-1}^{\infty}\left\{(x_0, x_{-1}) \in \mathbb{R}^2 : x_0 = -\frac{1}{a}\left(\frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+^n - \lambda_-^n}\right)x_{-1}\right\}. \tag{4}$$

The solution of equation (1) is

$$x_n = \frac{x_0x_{-1}(\lambda_+ - \lambda_-)}{[x_{-1}(\lambda_+^{n+1} - \lambda_-^{n+1}) + ax_0(\lambda_+^n - \lambda_-^n)]}, \quad n = -1, 0, 1, ... \tag{5}$$

Consider the set

$$S = F \cup D,$$

where $D = \{(u,v) \in \mathbb{R}^2 : au^2 + buv - v^2 = 0\}$. Note that for $(u,v) \in \mathbb{R}^2$, $c_2(u,v) = 0$ implies that $au^2 + buv - v^2 = 0$.

Theorem 2.1. *Assume that $(x_0, x_{-1}) \notin S$ and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1). Then the following statements are true.*

(1) *If $a + b < 1$, then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.*
(2) *If $a + b = 1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to $\frac{1}{c_2}$.*
(3) *If $a + b > 1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. When $a + b < 1$, we have that $\lambda_+ < 1$. But $|\lambda_-| < |\lambda_+|$, then

$$x_n = \frac{1}{c_1\lambda_-^n + c_2\lambda_+^n} = \frac{1}{\lambda_+^n(c_1(\frac{\lambda_-}{\lambda_+})^n + c_2)}.$$

That is

$$x_n \to \infty(sgn(c_2)) \quad \text{as} \quad n \to \infty,$$

from which (1) follows.

When $a + b = 1$, we have that $\lambda_+ = 1$. It follows that

$$x_n = \frac{1}{c_1\lambda_-^n + c_2} \to \frac{1}{c_2} \quad \text{as} \quad n \to \infty,$$

from which (2) follows.

When $a + b > 1$, we have that $\lambda_+ > 1$. Then

$$x_n = \frac{1}{\lambda_+^n(c_1(\frac{\lambda_-}{\lambda_+})^n + c_2)} \to 0 \quad \text{as} \quad n \to \infty,$$

from which (3) follows. $\square$

Theorem 2.2. *The subset*

$$D = \{(x,y) \in \mathbb{R}^2 : ax^2 + bxy - y^2 = 0\}$$

is an invariant subset of the set S.

Proof. Let $(x_0, x_{-1}) \in D$. We show that $(x_n, x_{n-1}) \in D$ for each $n \in N$. The proof is by induction on n. For $n = 0$, we have that

$$ax_0^2 + bx_0x_{-1} - x_{-1}^2 = 0.$$

This implies that

$$ax_0 + bx_{-1} = \frac{x_{-1}^2}{x_0}.$$

Now for $n = 1$, we have

$$ax_1^2 + bx_1x_0 - x_0^2 = a\frac{x_0^2x_{-1}^2}{(ax_0 + bx_{-1})^2} + bx_0\frac{x_0x_{-1}}{ax_0 + bx_{-1}} - x_0^2$$
$$= a\frac{x_0^4}{x_{-1}^2} + b\frac{x_0^3}{x_{-1}} - x_0^2 = \frac{x_0^2}{x_{-1}^2}(ax_0^2 + bx_0x_{-1} - x_{-1}^2) = 0.$$

This implies that $(x_1, x_0) \in D$.
Suppose that the relation is true for $n = k$. That is $(x_k, x_{k-1}) \in D$. Then

$$ax_k + bx_{k-1} = \frac{x_{k-1}^2}{x_k}.$$

That is

$$\begin{aligned} ax_{k+1}^2 + bx_{k+1}x_k - x_k^2 &= a\frac{x_k^2 x_{k-1}^2}{(ax_k + bx_{k-1})^2} + bx_k\frac{x_k x_{k-1}}{ax_k + bx_{k-1}} - x_k^2 \\ &= a\frac{x_k^4}{x_{k-1}^2} + b\frac{x_k^3}{x_{k-1}} - x_k^2 = \frac{x_k^2}{x_{k-1}^2}(ax_k^2 + bx_k x_{k-1} - x_{k-1}^2) = 0. \end{aligned}$$

Therefore, $(x_{k+1}, x_k) \in D$.
This completes the proof. □

Theorem 2.3. *Assume that $b = a - 1$ and let $(x_0, x_{-1}) \notin F$. If $c_2 = 0$, then $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period two.*

Proof. Clear that if $b = a - 1$, then $\lambda_- = -1$. It follows that

$$x_n = \frac{1}{c_1(-1)^n + c_2\lambda_+^n}.$$

But as $c_2 = 0$, we have that $x_0 = -x_{-1}$ and so $c_1 = \frac{1}{x_0}$. This implies that

$$x_n = \frac{1}{c_1(-1)^n} = \begin{cases} \frac{1}{c_1} = x_0, & n \text{ even}, \\ -\frac{1}{c_1} = -x_0, & n \text{ odd}. \end{cases}$$

This completes the proof. □

2.2. **Case $a < 0$, $b < 0$.** In this subsection, suppose that both a and b are negative. If $b^2 + 4a > 0$, the two roots λ_- and λ_+ are also negative such that $\lambda_- < \frac{b}{2} < \lambda_+ < 0$.

Theorem 2.4. *Assume that $(x_0, x_{-1}) \notin S$ and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1). Then the following statements are true.*

(1) *If $b < a - 1$, then $\{x_n\}_{n=-1}^{\infty}$ is converges to zero.*
(2) *If $b = a - 1$, then we have the following:*
   (a) *If $a \leq -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*
   (b) *If $a > -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to a period-2 solution.*
(3) *If $-2\sqrt{-a} > b > a - 1$, then we have the following:*
   (a) *If $a > -1$, then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.*
   (b) *If $a < -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. When $b < a - 1$, we have that $\lambda_- < -1 < \lambda_+ < 0$. Then

$$x_n = \frac{1}{\lambda_-^n(c_1 + c_2(\frac{\lambda_+}{\lambda_-})^n)} \to 0 \quad \text{as} \quad n \to \infty,$$

from which (1) follows.
When $b = a - 1$, we have that $\lambda = -1$ is a root of equation (3).
If $a = -1$, then $\lambda = -1$ is a repeated root and then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.
Now suppose that $a \neq -1$. If $a < -1$, then $\lambda_- = -1$ and $\lambda_+ = a$, from which (2a) follows. Similarly, when $a > -1$, $\lambda_- = a$ and $\lambda_+ = -1$, from which (2b) follows.
When $b > a - 1$, we have either $\lambda_- < \lambda_+ < -1$ or $-1 < \lambda_- < \lambda_+ < 0$.
If $a > -1$, then $-1 < \lambda_- < \lambda_+ < 0$ from which (3a) follows. If $a < -1$, then $\lambda_- < \lambda_+ < -1$ from which (3b) follows. This completes the proof. □

Theorem 2.5. *Assume that $a < 0$ and $b < 0$ and let $(x_0, x_{-1}) \notin F$. The following statements hold:*

(1) *If $b = a - 1$, then there exist periodic solutions of prime period two.*
(2) *All solutions of equation (1) oscillate about zero*

Proof. (1) Suppose that $b = a - 1$ and let $(x_0, x_{-1}) \notin F$. If $a > -1$ let Then

$$x_n = \frac{1}{c_1(-1)^n + c_2\lambda_+^n}.$$

If $c_2 = 0$, we have that $x_0 = -x_{-1}$. Then $c_1 = \frac{1}{x_0}$. This implies that

$$x_n = \frac{1}{c_1(-1)^n} = \begin{cases} \frac{1}{c_1} = x_0, & n \text{ even}, \\ -\frac{1}{c_1} = -x_0, & n \text{ odd}. \end{cases}$$

If $a < -1$, then

$$x_n = \frac{1}{c_1\lambda_-^n + c_2(-1)^n}.$$

If $c_1 = 0$, then $c_2 = \frac{1}{x_0}$ and therefore, $\{x_n\}_{n=-1}^{\infty}$ is periodic with prime period two.

(2) Clear that $a < 0$ and $b < 0$ implies negative roots for equation (3). Using Theorem (1.1), we get the result.

□

The rest of this subsection is devoted to discuss the case $b^2 + 4a \leq 0$.
When $b^2 + 4a = 0$ the solution of equation (2) is

$$y_n = c_1(\frac{b}{2})^n + c_2(\frac{b}{2})^n n, \quad n = -1, 0, 1, \ldots.$$

By a simple calculations, we can obtain the solution

$$x_n = \frac{x_0x_{-1}}{(\frac{b}{2})^n(-\frac{b}{2}x_0n + (1+n)x_{-1})}, \quad n = -1, 0, 1, \ldots. \tag{6}$$

The forbidden set F of equation (1) in this case is

$$F = \bigcup_{n=-1}^{\infty} \left\{(x_0, x_{-1}) \in \mathbb{R}^2 : x_0 = \frac{2(1+n)x_{-1}}{bn}\right\}.$$

Similarly, when $b^2 + 4a < 0$, the solution of equation (2) is

$$x_n = \frac{x_0 x_{-1} \sin\theta}{(-a)^{\frac{n}{2}}(x_{-1}\sin(n+1)\theta - \sqrt{-a}(x_0 \sin n\theta))}, \tag{7}$$

where $\theta = \tan^{-1}\frac{\sqrt{-b^2-4a}}{b} \in]\frac{\pi}{2}, \pi[$.

The forbidden set F of equation (1) in this case is

$$F = \bigcup_{n=-1}^{\infty}\left\{(x_0, x_{-1}) \in \mathbb{R}^2 : x_0 = \frac{x_{-1}\sqrt{-b^2-4a}}{-2a}\left(\frac{b}{\sqrt{-b^2-4a}} + \cot n\theta\right)\right\}.$$

Theorem 2.6. *Assume that $b = -2\sqrt{-a}$ and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. The following statements are true.*

1. *If $a \leq -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*
2. *If $a > -1$, then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. If $b = -2\sqrt{-a}$, then $\lambda = -\sqrt{-a}$ is a repeated root of equation (3). That is

$$x_n = \frac{1}{c_1(-a)^{\frac{n}{2}} + c_2(-a)^{\frac{n}{2}} n}.$$

When $a \leq -1$, we have that $(-a)^n n$ diverges to ∞ as $n \to \infty$. Then $\{x_n\}_{n=-1}^{\infty}$ converges to zero, from which (1) follows. When $a > -1$, we have that $(-a)^n n$ converges to zero, from which (2) follows. ☐

Theorem 2.7. *Assume that $0 > b > -2\sqrt{-a}$ and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. The following statements are true.*

1. *If $a < -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*
2. *If $a = -1$, then $\{x_n\}_{n=-1}^{\infty}$ is bounded.*
3. *If $a > -1$, then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. If $0 > b > -2\sqrt{-a}$, then the roots of equation (3) are complex and $|\lambda_{\pm}| = \sqrt{-a}$. Then

$$x_n = \frac{1}{(-a)^{\frac{n}{2}}(c_1 \cos n\theta + c_2 \sin n\theta)},$$

where $\theta = \tan^{-1}\frac{\sqrt{-b^2-4a}}{b} \in]\frac{\pi}{2}, \pi[$. When $a < -1$, we have that $(-a)^n$ diverges to ∞ as $n \to \infty$. Then $\{x_n\}_{n=-1}^{\infty}$ converges to zero, from which (1) follows. When $a = -1$, we have that

$$x_n = \frac{1}{c_1 \cos n\theta + c_2 \sin n\theta},$$

$\theta = \tan^{-1}\frac{\sqrt{-b^2+4}}{b}$. As $(x_0, x_{-1}) \notin F$, we have that for any $n \in \mathbb{N}$

$$c_1 \cos n\theta + c_2 \sin n\theta \neq 0.$$

This means that, there exists $\epsilon > 0$ such that $|c_1 \cos n\theta + c_2 \sin n\theta| > \epsilon$, from which (2) follows. When $a > -1$, we have that $(-a)^n$ converges to zero, from which (3) follows. □

Theorem 2.8. *Assume that $a = -1$ and $0 > b > -2$, and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin F$. If $\theta = \frac{p}{q}\pi$, where p and q are positive relatively prime integers such that $\frac{q}{2} < p < q$, then $\{x_n\}_{n=-1}^{\infty}$ is a periodic solution with prime period $2q$.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin F$. Clear that, if $a = -1$ and $-2 < b < 0$, the angle $\theta = \frac{p}{q}\pi \in]\frac{\pi}{2}, \pi[$.
The solution (7) becomes

$$x_n = \frac{x_0 x_{-1} \sin\theta}{(x_{-1}\sin(n+1)\theta - x_0 \sin n\theta)}.$$

Then

$$\begin{aligned} x_{n+2q} &= \frac{x_0 x_{-1} \sin\theta}{(x_{-1} sin(n+2q+1)\theta - x_0 \sin(n+2q)\theta)} \\ &= \frac{x_0 x_{-1} \sin\theta}{(x_{-1}\sin((n+1)\theta + 2q\theta) - x_0 \sin(n\theta + 2q\theta)} \\ &= \frac{x_0 x_{-1} \sin\theta}{(x_{-1}\sin((n+1)\theta + 2p\pi) - x_0 \sin(n\theta + 2p\pi)} \\ &= x_n. \end{aligned}$$

This completes the proof. □

Example (1). Figure 1. shows that if $a = 0.2$, $b = 0.8$ $(a + b = 1)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = -0.5$ and $x_0 = 1$ converges to $\frac{1}{c_2} = 2$.

Example (2). Figure 2. shows that if $a = 2$, $b = 1$, $(b = a - 1)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = -2$ and $x_0 = 2$ $(c_2 = 0)$ is periodic with prime period two.

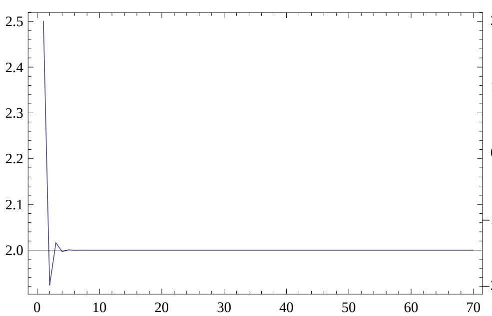

FIGURE 1. $x_{n+1} = \frac{x_n x_{n-1}}{0.2x_n + 0.8x_{n-1}}$.

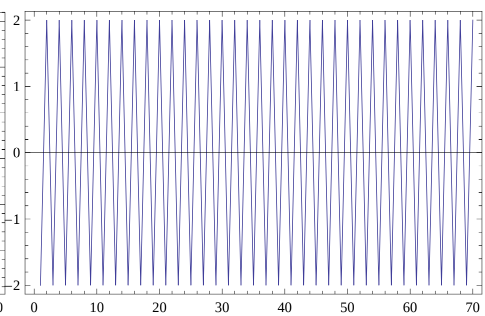

FIGURE 2. $x_{n+1} = \frac{x_n x_{n-1}}{2x_n + x_{n-1}}$.

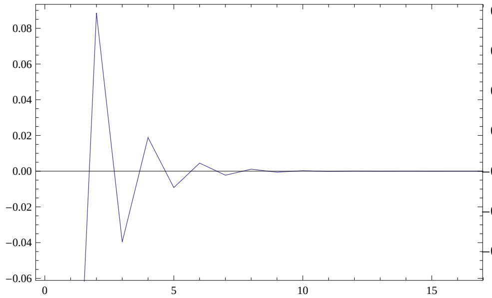

FIGURE 3. $x_{n+1} = \frac{x_n x_{n-1}}{-2x_n - 3x_{n-1}}$.

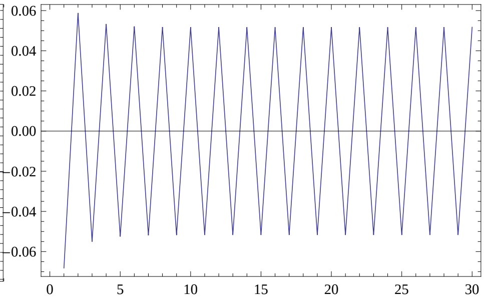

FIGURE 4. $x_{n+1} = \frac{x_n x_{n-1}}{-0.5x_n - 1.5x_{n-1}}$.

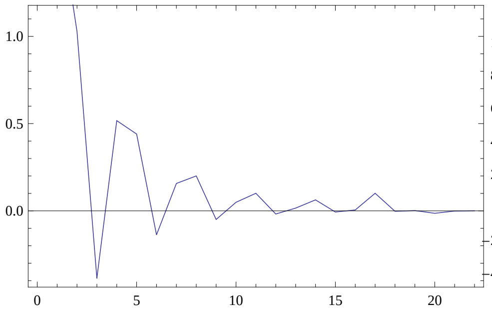

FIGURE 5. $x_{n+1} = \frac{x_n x_{n-1}}{-2x_n - 1.5x_{n-1}}$.

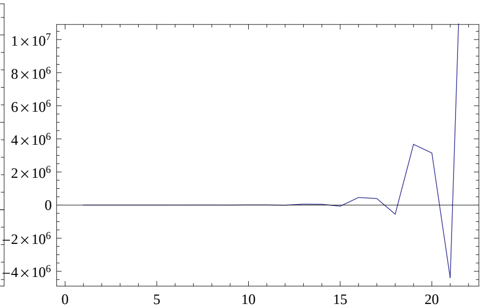

FIGURE 6. $x_{n+1} = \frac{x_n x_{n-1}}{-0.25x_n - 0.5x_{n-1}}$.

Example (3). Figure 3. shows that if $a = -2$, $b = -3$ $(b = a - 1)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 1.1$ and $x_0 = 1.2$ converges to zero.

Example (4). Figure 4. shows that if $a = -0.5$, $b = -1.5$, $(b = a - 1)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = -1.5$ and $x_0 = 0.1$ converges to a period-2 solution.

Example (5). Figure 5. shows that if $a = -2$, $b = -1.5$ $(0 > b > -2\sqrt{-a} \approx -2.8284$ and $a < -1)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 2.5$ and $x_0 = -1.1$ converges to zero.

Example (6). Figure 6. shows that if $a = -0.25$, $b = -0.5$, $0 > b > -2\sqrt{-a} = -1$ and $a > -1)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 1.5$ and $x_0 = -2.1$ is unbounded.

Example (7). Figure 7. shows that if $a = -1$, $b = -\frac{5-\sqrt{5}}{2}$ $(b^2 + 4a < 0)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 2.7$ and $x_0 = -0.4$ is periodic with prime period 20. Note that $\theta = \frac{7}{10}\pi$.

Example (8). Figure 8. shows that if $a = -1$, $b = -\frac{3-\sqrt{5}}{2}$ $(b^2 + 4a < 0)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = -2.2$ and $x_0 = -0.5$ is periodic with prime period 10. Note that $\theta = \frac{3}{5}\pi$.

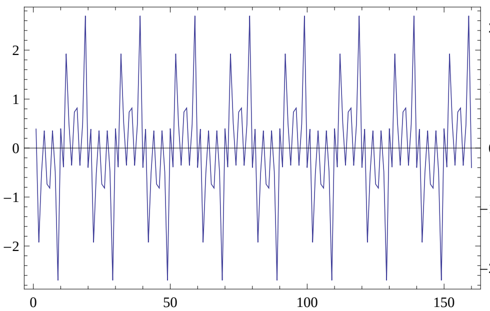

FIGURE 7. $x_{n+1} = \frac{x_n x_{n-1}}{-x_n - \frac{5-\sqrt{5}}{2} x_{n-1}}$.

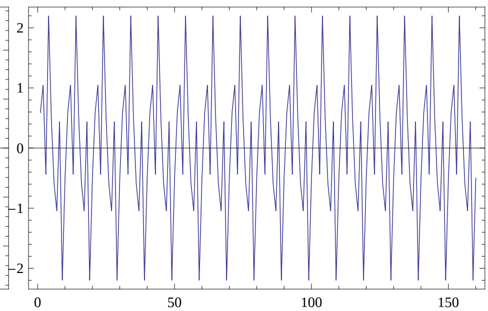

FIGURE 8. $x_{n+1} = \frac{x_n x_{n-1}}{-x_n - \frac{3-\sqrt{5}}{2} x_{n-1}}$.

3. CASE $ab < 0$

3.1. **Case** $a > 0$ **and** $b < 0$. In this case $b^2 + 4a > 0$, that is the two roots λ_- and λ_+ are real such that $\lambda_- < \frac{b}{2} < 0 < \lambda_+$ and $|\lambda_-| > |\lambda_+|$.

Theorem 3.1. *Assume that* $(x_0, x_{-1}) \notin S$ *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation (1). The following statements are true.*

(1) *If* $b < a - 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is converges to zero.*
(2) *If* $b = a - 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *converges to a period-2 solution.*
(3) *If* $b > a - 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is unbounded.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. When $b < a - 1$, we have that $\lambda_- < -1$. It follows that

$$x_n = \frac{1}{\lambda_-^n (c_1 + c_2 (\frac{\lambda_+}{\lambda_-})^n)} \to 0 \quad \text{as} \quad n \to \infty,$$

from which (1) follows.
When $b = a - 1$, we have that $\lambda_- = -1$. Then x_n converges to the period-2 solution $\{..., \frac{1}{c_1}, -\frac{1}{c_1}, \frac{1}{c_1}, -\frac{1}{c_1}, ...\}$, from which (2) follows.
When $b > a - 1$, we have that $0 > \lambda_- > -1$. If $a > -1$, then $-1 < \lambda_- < \lambda_+ < 0$ from which (3) follows.
This completes the proof. □

Theorem 3.2. *Assume that* $b = a - 1$. *Then there exist periodic solutions of prime period two.*

Proof. Assume that $b = a - 1$ and let $(x_0, x_{-1}) \notin F$. It is sufficient to see that if c_2=0, then $c_1 = \frac{1}{x_0}$ and the result follows. □

3.2. **Case $a < 0$ and $b > 0$.** In this case, $b^2 + 4a$ can be negative, zero or a positive real number. If $b^2 + 4a > 0$, then the roots of equation (3) are positive such that $0 < \lambda_- < \lambda_+$.

Theorem 3.3. *Assume that $(x_0, x_{-1}) \notin S$ and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1). The following statements are true.*

(1) *If $b > 1 - a$, then $\{x_n\}_{n=-1}^{\infty}$ is converges to zero.*
(2) *If $b = 1 - a$, then we have the following:*
  (a) *If $a > -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to $\frac{1}{c_2}$.*
  (b) *If $a < -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*
(3) *If $2\sqrt{-a} < b < 1 - a$, then we have the following:*
  (a) *If $a > -1$, then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.*
  (b) *If $a < -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. When $b > 1 - a$, we have that $\lambda_- < 1 < \lambda_+$. It follows that

$$x_n = \frac{1}{\lambda_+^n (c_1 (\frac{\lambda_-}{\lambda_+})^n + c_2)} \to 0 \quad \text{as} \quad n \to \infty,$$

from which (1) follows.
When $a + b = 1$, $\lambda = 1$ is a root for equation (3).
If $a > -1$, then $\lambda_- = -a$ and $\lambda_+ = 1$. Now

$$x_n = \frac{1}{(c_1 \lambda_-^n + c_2)} \to \frac{1}{c_2} \quad \text{as} \quad n \to \infty,$$

from which (2a) follows. If $a < -1$, then $\lambda_- = 1$ and $\lambda_+ = -a$, from which (2b) follows.
Note that, when $a + b = 1$, $b^2 + 4a > 0$ when $a \neq -1$.
When $2\sqrt{-a} < b < 1 - a$, we have either $0 < \lambda_- < \lambda_+ < 1$ or $1 < \lambda_- < \lambda_+$.
If $a > -1$, we have that $0 < \lambda_- < \lambda_+ < 1$, from which (3a) follows.
If $a < -1$, then $1 < \lambda_- < \lambda_+$, from which (3b) follows.
This completes the proof. □

Theorem 3.4. *Assume that $0 < b < 2\sqrt{-a}$ and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. The following statements are true.*

(1) *If $a < -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to zero.*
(2) *If $a = -1$, then $\{x_n\}_{n=-1}^{\infty}$ is bounded.*
(3) *If $a > -1$, then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin S$. It is sufficient to see that, if $0 < b < 2\sqrt{-a}$, then the roots of equation (3) are complex such that $|\lambda_\pm| = \sqrt{-a}$ and $\theta = tan^{-1}\frac{\sqrt{-b^2-4a}}{b} \in]0, \frac{\pi}{2}[$. The rest of the proof is similar to that of Theorem (2.7) and will be omitted. □

Theorem 3.5. *Assume that $a = -1$ and $0 > b > -2$, and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}) \notin F$. If $\theta = \frac{p}{q}\pi$, where p and*

q are positive relatively prime integers such that $0 < p < \frac{q}{2}$, then $\{x_n\}_{n=-1}^{\infty}$ is a periodic solution with prime period $2q$.

Proof. The proof is similar to that of Theorem (2.8) and will be omitted. $\square$

Example (9). Figure 9. shows that if $a = 0.9$, $b = -0.1$ $(b = a - 1)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 2.5$ and $x_0 = 2.1$ converges to period-2 solution.

Example (10). Figure 10. shows that if $a = -0.25$, $b = 1.1$, $(1 = 2\sqrt{-a} < b < 1 - a = 1.25)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 2.5$ and $x_0 = -1.1$ is unbounded.

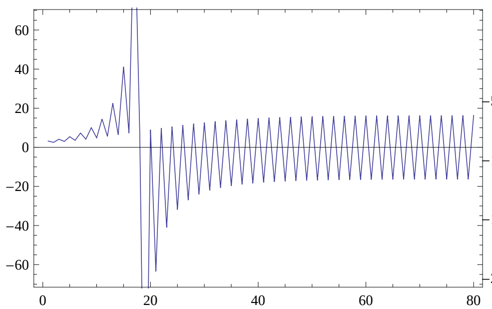

FIGURE 9. $x_{n+1} = \frac{x_n x_{n-1}}{0.9x_n - 0.1x_{n-1}}$.

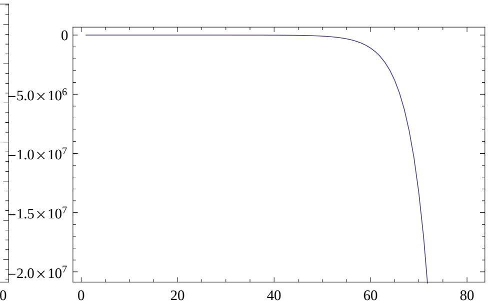

FIGURE 10. $x_{n+1} = \frac{x_n x_{n-1}}{-0.25x_n + 1.1x_{n-1}}$.

Example (11). Figure 11. shows that if $a = -1$, $b = -1$ $(b^2 + 4a < 0)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 2.5$ and $x_0 = -1.1$ is a period-3 solution.

Example (12). Figure 12. shows that if $a = -1$, $b = 1$, $(b^2 + 4a < 0)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 2.5$ and $x_0 = -1.1$ is a period-6 solution.

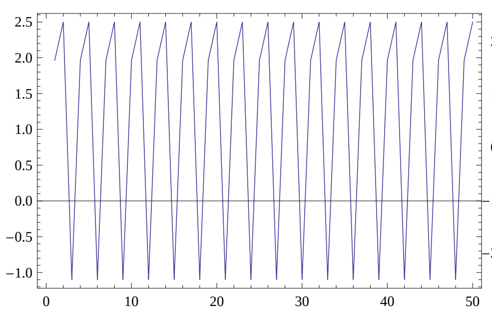

FIGURE 11. $x_{n+1} = \frac{x_n x_{n-1}}{-x_n - x_{n-1}}$.

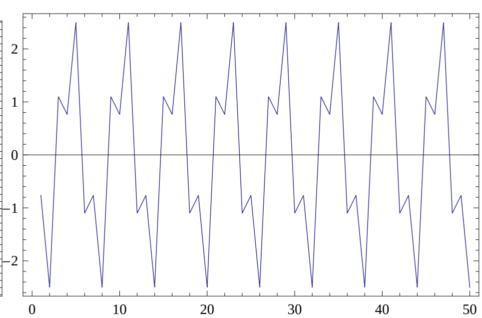

FIGURE 12. $x_{n+1} = \frac{x_n x_{n-1}}{-x_n + x_{n-1}}$.

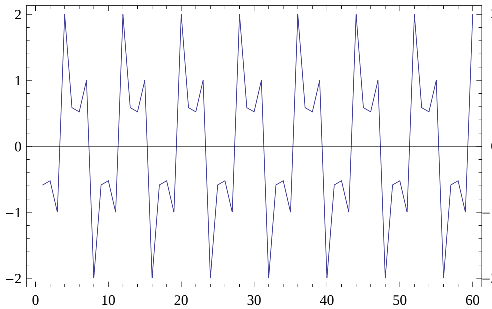

FIGURE 13. $x_{n+1} = \frac{x_n x_{n-1}}{-x_n + \sqrt{2}x_{n-1}}$.

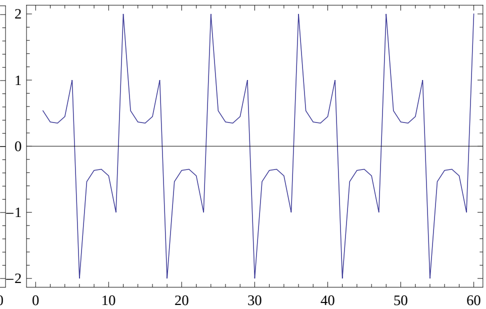

FIGURE 14. $x_{n+1} = \frac{x_n x_{n-1}}{-x_n + \sqrt{3}x_{n-1}}$.

Example (13). Figure 13. shows that if $a = -1$, $b = \sqrt{2}$ $(b^2 + 4a < 0)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = 1$ and $x_0 = -2$ is periodic with prime period 8. Note that $\theta = \frac{1}{4}\pi$.

Example (14). Figure 14. shows that if $a = -1$, $b = \sqrt{3}$ $(b^2 + 4a < 0)$. Then the solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) with initial conditions $x_{-1} = -1$ and $x_0 = 2$ is periodic with prime period 12. Note that $\theta = \frac{1}{6}\pi$.

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